§ Linear transformations.

Def: Let V and W be vector spaces over F.

A linear transformation from V to W. is a map

$$T: V \longrightarrow W.$$

s.t. (a). $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

(b) $T(a\vec{x}) = aT(\vec{x})$

for all $\vec{x}, \vec{y} \in V$ and $a \in F$.

Quick consequences:

Let $T: V \to W$ be a linear transformation. Then

(i) $T(\vec{0}_V) = \vec{0}_W$.

(ii) $T(a_1\vec{v_1} + \cdots + a_n\vec{v_n}) = a_1 T(\vec{v_1}) + \cdots + a_n T(\vec{v_n})$ $\forall\, \vec{v_1}, \cdots, \vec{v_n} \in V$, $a_1 \cdots a_n \in F$.

(i.e., T preserves linear combinations)

pf: (i) $\cancel{T(\vec{0}_V)} = T(\vec{0}_V + \vec{0}_V) = \cancel{T(\vec{0}_V)} + T(\vec{0}_V) \in W$

$\Rightarrow T(\vec{0}_V) = \vec{0}_W$ by cancellation law in W.

(ii) By induction on n.

Examples :

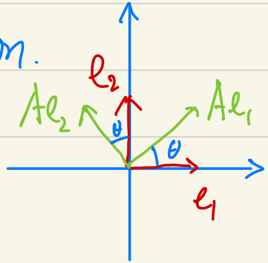
- Let $A \in M_{m\times n}(F)$. If we regard F^n and F^m as space of column vectors;

  then $L_A : F^n \longrightarrow F^m$ defined by $L_A(\vec{x}) := A\vec{x}$ is linear.

  This is called the left multiplication by A.

  e.g. Left multiplication by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ gives the rotation.

  $T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by θ in the counterclockwise direction.



- $T: P_n(\mathbb{R}) \longrightarrow P_{n-1}(\mathbb{R})$ defined by $T(f(x)) := f'(x)$ is a linear transformation. $\left(\begin{array}{l} T(f+g) = (f+g)' = f' + g' = T(f) + T(g) \\ T(af) = (a \cdot f)' = a \cdot f' = a \cdot T(f) \end{array} \right)$

- $T: P_{n-1}(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$ defined by $T(f(x)) = \int_0^x f(t)\,dt$ is a linear transformation.

- Zero transformation $T_0: V \longrightarrow W$ defined by $T_0(\vec{x}) = \vec{0}_W \quad \forall \vec{x} \in V$

  Identity transformation. $I_V: V \longrightarrow V$ defined by $I_V(\vec{x}) = \vec{x} \quad \forall \vec{x} \in V$.

§ Null spaces and ranges.

Def: Let V and W be vector spaces and $T: V \to W$ be linear.

The null space / kernel of T is defines as

$$N(T) := \{\vec{x} \in V : T(\vec{x}) = \vec{0}\} \subset V.$$

The range / image of T is defined as

$$R(T) := \{T(\vec{x}) : \vec{x} \in V\} \subset W.$$

Example:

- For the left multiplication $L_A : F^n \longrightarrow F^m$ by a matrix $A \in M_{m\times n}(F)$.

  $N(L_A) = N(A)$, the null space of A.

  $R(L_A) = \mathcal{C}(A)$, the column space of A

  i.e., space of linear combination of column vectors of A.

$$L_A(\vec{x}) = A \cdot \vec{x} = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{matrix} | \\ v_1 \\ | \end{matrix} + \cdots + x_n \begin{matrix} | \\ v_n \\ | \end{matrix}.$$

- For $T: \overset{\text{dim}=n+1}{P_n(\mathbb{R})} \longrightarrow P_{n-1}(\mathbb{R})$ defined by $T(f(x)) := f'(x)$.

  $N(T) = \{a_0 \in P_n(\mathbb{R}) : a_0 \in \mathbb{R}\} = P_0(\mathbb{R})$.   dim $= 1$

  $R(T) = P_{n-1}(\mathbb{R})$ dim $= n$ $\{1, x, \cdots, x^{n-1}\}$

- Identity map $I_V: V \longrightarrow V$. $N(I_V) = \{0_V\}$, $R(I_V) = V$

  Zero map. $T_0: V \longrightarrow W$. $N(T_0) = V$, $R(T_0) = \{\vec{0}_W\}$

- Surjective / onto $T: V \longrightarrow W$ $\Leftrightarrow$ $\boxed{R(T) = W}$.

  injective / one-to-one $T: V \longrightarrow W$ s.t. $T(\vec{x}) \neq T(\vec{y})$ for $\vec{x} \neq \vec{y}$,

  $\Longleftrightarrow$ $\boxed{N(T) = \{\vec{0}_V\}}$

$(\Rightarrow)$ $T(\vec{0}_V) = \vec{0}_W$, and unique; $(\Leftarrow)$: $T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) \neq 0$ if $\vec{x} \neq \vec{y}$

Proposition: Let $T: V \to W$ be a linear transformation.

Then $N(T)$ and $R(T)$ are subspace of V and W respectively.

Pf: Since $T(\vec{0}_V) = \vec{0}_W$, we have $\vec{0}_V \in N(T)$ and $\vec{0}_W \in R(T)$.

Let $\vec{x}, \vec{y} \in N(T)$ and $a \in F$. Then

$$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) = \vec{0}_W + \vec{0}_W = \vec{0}_W$$
$$T(a\vec{x}) = a \cdot T(\vec{x}) = a \cdot \vec{0}_W = \vec{0}_W.$$

$\Rightarrow \vec{x}+\vec{y} \in N(T)$ and $a\vec{x} \in N(T)$. Hence $N(T) \subset V$ is a subspace.

Now let $\vec{u}, \vec{v} \in R(T)$ and $a \in F$.

Then there exist $\vec{x}, \vec{y} \in V$ s.t. $T(\vec{x}) = \vec{u}$ and $T(\vec{y}) = \vec{v}$.

So $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) = \vec{u} + \vec{v}$. $\Rightarrow \vec{u}+\vec{v} \in R(T)$

and $T(a\vec{x}) = a \cdot T(\vec{x}) = a \cdot \vec{u} \Rightarrow a \cdot \vec{u} \in R(T)$.

Hence $R(T)$ is a subspace of W.

$\square$

Proposition: Let $T: V \to W$ linear transformation.

If $\beta = \{\vec{v_1}, \dots, \vec{v_n}\}$ is a basis for V.

Spanning set.

then $R(T) = \mathrm{span}(T(\beta)) = \mathrm{span}\{T(\vec{v_1}), \dots, T(\vec{v_n})\}$

Pf: • Since $T(\vec{v_j}) \in R(T)$ for $j = 1, \dots, n$ and $R(T)$ is a subspace.

we have $\mathrm{span}\{T(\vec{v_1}), \dots, T(\vec{v_n})\} \subset R(T)$ ✓

• Conversely, let $\vec{w} = T(\vec{x}) \in R(T)$ where $\vec{x} \in V$.

Then $\exists\, a_1, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{j=1}^{n} a_j \vec{v_j}$

So $\vec{w} = T(\vec{x}) = T\left(\sum_{j=1}^{n} a_j \vec{v_j}\right) = \sum_{j=1}^{n} a_j T(\vec{v_j}) \in \mathrm{span}\{T(\vec{v_1}), \dots, T(\vec{v_n})\}$

This proves the reverse inclusion $R(T) \subset \mathrm{span}\{T(\vec{v_1}), \dots, T(\vec{v_n})\}$ ✓ $\square$.

Next Goal: "Measure" the size of subspace $N(T)$ & $R(T)$

Intuitively, the larger $N(T)$, the smaller $R(T)$

the smaller $N(T)$, the larger $R(T)$.

Def: Let $T: V \to W$ linear transformation, If $N(T)$ and $R(T)$ are finite-dimension.

Define $\boxed{\text{nullity}(T) := \dim N(T)}$

$\boxed{\text{rank}(T) := \dim R(T)}$

Theorem (Rank-Nullity Theorem)

Let V, W be vector spaces. St. V is finite-dimensional.

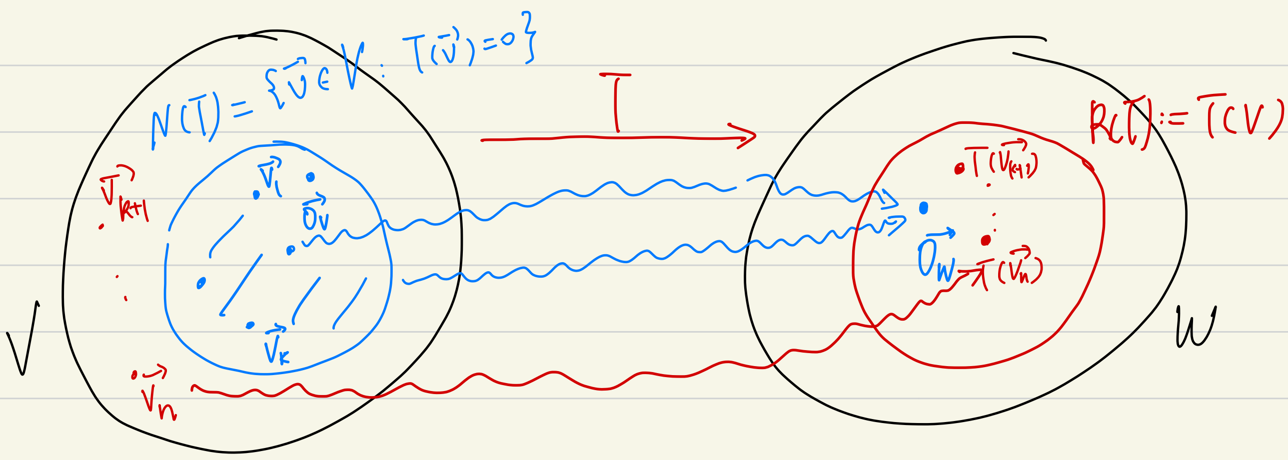
Then for any linear transformation $T: V \longrightarrow W$, we have

$$\underbrace{\text{nullity}(T)}_{\dim N(T)} + \underbrace{\text{rank}(T)}_{\dim R(T)} = \dim V.$$

Pf: Let $\dim V = n$. and $\dim N(T) = k \le n$.

Choose a basis $\{\vec{v}_1, \cdots, \vec{v}_k\}$ for $N(T)$

Lin indep $\Rightarrow$ Can extend to a basis $\beta = \{\vec{v}_1, \cdots, \vec{v}_k, \vec{v}_{k+1}, \cdots, \vec{v}_n\}$ of V.



Rank-Nullity Theorem: $\dim \underbrace{N(T)}_{}^{=k} + \dim R(T) = \dim V^{=n}$.

$\dim R(T) \overset{?}{=} n-k$

Claim: $S = \{T(\vec{v}_{k+1}), \cdots, T(\vec{v}_n)\}$ is a basis for $R(T)$.

- S span: We proved last time $R(T) = \text{Span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$

(Since $T(\vec{v}_1) = \cdots = T(\vec{v}_k) = 0$) $= \text{span}\{T(\vec{v}_{k+1}), \cdots, T(\vec{v}_n)\}$

$= \text{span } S$.

- S lin. indep: Suppose $\exists\, b_{k+1}, \cdots, b_n \in F$ s.t. $\sum_{i=k+1}^{n} b_i T(\vec{v_i}) = 0$

Since T is linear, $T\left(\sum_{i=k+1}^{n} b_i \vec{v_i}\right) = \sum_{i=k+1}^{n} b_i T(\vec{v_i}) = \vec{0}$.

$$\Rightarrow \sum_{i=k+1}^{n} b_i \vec{v_i} \in N(T).$$

Thus, $b_{k+1}\vec{v_{k+1}} + \cdots + b_n \vec{v_n} = c_1\vec{v_1} + \cdots + c_k \vec{v_k}$ for some $c_i \in F$

As $\beta = \{\vec{v_1}, \cdots, \vec{v_n}\}$ is a basis for V.
we must have $b_{k+1} = \cdots = b_n = 0$. (and $c_1 = \cdots = c_k = 0$).

Hence, S is lin. indep and thus forms a basis for $R(T)$

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Corollary: Suppose V and W are vector spaces of equal finite dimensions. $T: V \to W$ is a linear transformation.

Then TFAE:

(a). T is injective

(b). T is surjective

(c) $\operatorname{rank}(T) = \dim V$.

(MATH 1030: $M_{n \times n}$ invertible $\Leftrightarrow \operatorname{rank} M = n$)

Pf: T is injective

$\Leftrightarrow N(T) = \{\vec{0}\} \Leftrightarrow \dim N(T) = 0$.

$\Leftrightarrow \operatorname{rank}(T) = \dim R(T) = \dim V$. (Rank-Nullity Thm)

$\Leftrightarrow \dim R(T) = \dim W$ (dim $V = \dim W$ by assumption)

$\Leftrightarrow R(T) = W$, i.e., T is surjective. $\square$

Remark: Corollary is <u>not</u> true in infinite-dimensional case.

e.g. $T_1: P(\mathbb{R}) \longrightarrow P(\mathbb{R})$

$f(x) \rightsquigarrow f'(x)$ surjective, but not injective

$T_2: P(\mathbb{R}) \longrightarrow P(\mathbb{R})$

$f(x) \rightsquigarrow \int_0^x f(t)\,dt$. injective but not surjective

Applications :

Example 1: $T: P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by

$\{1, x, x^2\}$

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt \qquad \text{Linear}$$

We have $R(T) = \operatorname{Span}\{T(1), T(x), T(x^2)\}$

$$= \operatorname{Span}\{3x,\ 2+\tfrac{3}{2}x^2,\ 4x+x^3\} \quad \text{lin. indep.}$$

$\Rightarrow$ $\operatorname{rank}(T) = \dim R(T) = 3$.

$\Rightarrow$ $\operatorname{nullity}(T) = \dim N(T) = 0$.

So we conclude that T is injective.

Example 2: Show that $\forall q(x) \in P(\mathbb{R})$ $\exists p(x) \in P(\mathbb{R})$ s.t. $[(x^2+5x+7)\,p(x)]'' = q(x)$.

Pf: Define a map $T: P(\mathbb{R}) \longrightarrow P(\mathbb{R})$ is linear (Check!)

$p(x) \rightsquigarrow [(x^2+5x+7)\,p(x)]''$

The original statement $\iff$ T is surjective.

Note: $(a_n x^n + \cdots + a_1 x + a_0)'' = n\cdot(n-1)\,a_n x^{n-2} + \cdots \neq 0$ if $a_n \neq 0$, $n \geq 2$.

$\Rightarrow T$ injective. BUT $P(\mathbb{R})$ inf-dim.

Instead of considering $P(\mathbb{R})$, we restrict to $T_n: P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$

Now, Corollary implies T_n is injective $\iff$ T_n is surjective $\forall n$

$\iff$ T is surjective.

$\square$

Next: Explicitly describe $T: V \to W$ $\rightsquigarrow$ Matrix Representation

Theorem: Let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ be a basis for V

Then given any $\vec{w}_1, \cdots, \vec{w}_n \in W$. $\exists!$ linear transformation

$T: V \to W$. s.t. $\boxed{T(\vec{v}_i) = \vec{w}_i}$ $\forall i = 1, \cdots, n$.

Pf: For $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i = a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n \in V$ (basis $\Rightarrow$ unique expression)

let $T(\vec{x}) = \sum a_i \vec{w}_i = a_1 \vec{w}_1 + \cdots + a_n \vec{w}_n \in W$

- $T(\vec{v_i}) = \vec{w_i} \quad \forall\, i = 1, \cdots, n$
- T is linear : For $\vec{x} = \sum_{i=1}^{n} a_i \vec{v_i}$, $\vec{y} = \sum_{i=1}^{n} b_i \vec{v_i} \in V$, $c \in F$

$$T(\vec{x} + \vec{y}) = T\left(\sum_{i=1}^{n} (a_i + b_i) \vec{v_i}\right) = \sum_{i=1}^{n} (a_i + b_i) \vec{w_i} = T(\vec{x}) + T(\vec{y}).$$

$$T(c\vec{x}) = T\left(\sum_{i=1}^{n} c a_i \vec{v_i}\right) = \sum_{i=1}^{n} c a_i \vec{w_i} = c \cdot T(\vec{x}).$$

- T is unique : Suppose $U : V \to W$ is linear s.t. $U(\vec{v_i}) = \vec{w_i}$.

Then $\forall\, \vec{x} = \sum_{i=1}^{n} a_i \vec{v_i} \in V$.

$$U(\vec{x}) = \sum_{i=1}^{n} a_i U(\vec{v_i}) = \sum_{i=1}^{n} a_i \vec{w_i} = T(\vec{x}) \quad \Leftrightarrow \quad U = T \quad \square.$$

Corollary: Let V be a vector space with a finite basis $\beta = \{\vec{v}_1, \cdots, \vec{v}_n\}$.

Then any linear transformation from V to another vector space W.

is completely determined by its values on β.

i.e., if $U, T: V \longrightarrow W$ are linear

& $U(\vec{v}_i) = T(\vec{v}_i) \quad \forall\, i = 1, \cdots, n$

then $U = T$.